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A PRIORI ESTIMATES AND EXISTENCE RESULTS FOR SOME SEMILINEAR ELLIPTIC PROBLEMS

P. L. Lions

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July 1979

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ABSTRACT

We study the existence of non-trivial, positive solutions of

$$(1)\qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

We prove the existence of a priori estimates for positive solutions of (1), provided f does not increase too rapidly at infinity and provided Ω has some geometrical properties. From these a priori estimates we deduce the existence of non-trivial, positive solutions of (1).

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AMS (MOS) Subject Classifications: 35B45, 35B99

Key Words: Semilinear problems, Symmetry, Non-trivial solutions

Work Unit Number 1 (Applied Analysis)

- A -

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SIGNIFICANCE AND EXPLANATION

Problems arising in, for example, physics, biology and chemical reactors lead to boundary value problems for semilinear elliptic equations. The question of identifying for which nonlinearities these semilinear problems have solutions is by no means settled. Roughly speaking, nonlinearities which grow too rapidly yield unsolvable problems while under other conditions on the growth existence can be proved. Here we improve some conditions which were previously known to guarantee the existence of positive solutions of such problems.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A PRIORI ESTIMATES AND EXISTENCE RESULTS FOR SOME SEMILINEAR ELLIPTIC PROBLEMS

P. L. Lions*

I. Introduction:

In this paper we consider the following problem: find a solution of u

$$(1) \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is a given nonlinear function such that $f(0) = 0$, and Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary. Obviously zero is a solution of (1), so we look for a non-trivial solution of (1), and actually we shall restrict ourselves to positive solutions.

Under various conditions on f, the existence of a positive, non-trivial solution to (1) has been proved, for instance, by P. H. Rabinowitz [9], [10], A. Ambrosetti and P. H. Rabinowitz [8], R. E. L. Turner [11], H. Brézis and R. E. L. Turner [4], R. Nussbaum [7], H. Amann [1], S. I. Pohozaev [8].

We shall be concerned here with the superlinear case i.e., we shall assume that the function f satisfies the condition

$$(2) \qquad \lim_{t\to\infty} \frac{f(t)}{t} > \lambda_1 ,$$

where λ_1 is the first eigenvalue of $-\Delta$, with Dirichlet boundary conditions. One way to solve (1), in this case, is to obtain L^∞ a priori estimates for positive solutions. This was done in [11], [4], assuming that

$$\lim_{t\to\infty} f(t)t^{-\frac{N+1}{N-1}} = 0 .$$

We prove here that such estimates can be obtained if

$$(3) \qquad \lim_{t\to\infty} f(t)t^{-\beta} = 0 \quad \text{for some} \quad \beta < \frac{N}{N-2} \quad (\text{if } N = 2 \text{ for any } \beta)$$

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provided Ω (when $N \geq 3$) has some geometrical properties. For example, we proved that there exists a constant C such that if u satisfies

$$-\Delta u = u^{\beta} \text{ in } \Omega, \quad u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

with $\beta < \frac{N}{N-2}$ and Ω convex, then $\|u\|_{L^{\infty}(\Omega)} \leq C$.

The main new ingredient used to obtain such estimates is the convenient application of some recent remarkable results of B. Gidas, Wei-Ming Ni and L. Nirenberg [5]. They prove that if u is a positive solution of (1), then the set of critical points of u is contained in a proper subset of Ω (independent of u and f), if Ω (for $N \geq 3$) has some geometrical properties. Using this result, an easy method is then used to give the needed a priori estimates and the existence of positive solutions of (1).

The author wishes to thank Profs. M. G. Crandall, B. Gidas, L. Nirenberg and R. E. L. Turner for various discussions on related topics.

II. A priori estimates:

Let Ω be a bounded, connected domain with a smooth boundary:

Theorem 1: We assume that f is locally Lipschitz from $\mathbb{R}_+$ to $\mathbb{R}_+$, satisfies (2) and (3). We also assume that if $N \geq 3$, Ω is convex. Then there exists a constant C such that for all $u \in C^2(\bar{\Omega})$ satisfying (1')

$$(1') \quad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u \geq 0 \text{ in } \Omega, & u = 0 \text{ on } \partial\Omega; \end{cases}$$

we have

$$\|u\|_{L^{\infty}(\Omega)} \leq C .$$

Remark 1: Instead of assuming Ω convex, we could have assumed

$$(*) \quad \begin{cases} \exists \Omega' \text{ proper closed subset of } \Omega \text{ such that for all positive } u \text{ satisfying} \\ \text{(1') the set } \{x \in \Omega \;\; \nabla u(x) = 0\} \subset \Omega' . \end{cases}$$

In view of the results of B. Gidas, Wei-Ming Ni and L. Nirenberg [5], (*) is satisfied for every domain Ω in $\mathbb{R}^2$, and for convex domains in $\mathbb{R}^N$ $(N \geq 3)$. Other cases

are given in [5]. The question of whether (•) is satisfied for any domain in $\mathbb{R}^N$ ($N \geq 3$) is open and a positive answer would imply that Theorem 1 is true without the restriction Ω convex.

Proof of Theorem 1: The proof is divided into two steps: 1) we prove that u and f are bounded in $L^1_{loc}(\Omega)$, 2) we deduce the $L^\infty(\Omega)$ estimate.

1) Let σ_1 be a fixed eigenfunction of $-\Delta$ corresponding to the first eigenvalue i.e.:

$$-\Delta\sigma_1 = \lambda_1\sigma_1 \text{ in } \Omega, \quad \sigma_1 = 0 \text{ on } \partial\Omega, \quad \sigma_1 \in C^2(\bar{\Omega}) .$$

We may choose σ_1 so that $\sigma_1 > 0$ in Ω. Now (as in [11], [4], [7], [3]) we multiply (1) by σ_1 and we obtain

$$\lambda_1 \int_\Omega u\sigma_1 dx = \int_\Omega f(u)\sigma_1 dx .$$

But using (2) there exists $\lambda > \lambda_1$, $C > 0$ such that $f(t) \geq \lambda t - C$ for every $t \geq 0$. Thus we deduce $\int_\Omega u\sigma_1 \leq$ const., and $\int_\Omega f(u)\sigma_1 dx \leq$ const. Since $\sigma_1 > 0$ in Ω, we have an estimation of u and $f(u)$ in $L^1_{loc}(\Omega)$.

2) Thus, $-\Delta u$ and u are in a bounded subset of $L^1_{loc}(\Omega)$. But this implies by standard regularity results (localize and use $L^1 \subset W^{-\varepsilon,p} \forall p < \frac{N}{N-\varepsilon}$) that u is bounded in $L^p_{loc}(\Omega)$ ($\forall p < \frac{N}{N-2}$ if $N \geq 3$, $\forall p < \infty$ if $N = 2$). Now we use (3) and we have

$$f(u) \text{ is bounded in } L^{q_1}_{loc}(\Omega) \text{ with } q_1 > 1 .$$

This implies u is bounded in $W^{2,q_1}_{loc}(\Omega)$ and by a standard bootstrap argument we obtain after a finite number of iterations of the preceding argument

$$u \text{ is bounded in } W^{2,q}_{loc}(\Omega) \text{ with } q > \frac{N}{2} ,$$

in particular u is bounded in $L^\infty_{loc}(\Omega)$. To conclude, we apply the previously mentioned result of [5]: indeed, there exists some open set Ω' (independent of f and u) such that if $u \not\equiv 0$

$$\{x \in \Omega \ \nabla u(x) = 0\} \subset \Omega' \subset \bar{\Omega}' \subset \Omega .$$

Thus, $\|u\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(\Omega')}$, and $\|u\|_{L^\infty(\Omega')} \le$ const.

Remark 2: 1) If $N = 2$, with a slight refinement of the second part of the proof, we can prove Theorem 1 with the assumption

$$\lim_{t\to\infty} f(t)\exp\{-t/C\} = 0\ , \tag{3'}$$

where C is some constant depending only on Ω, and on $\lim\limits_{t\to\infty} \frac{f(t)}{t}$. Indeed, if u is such that $-\Delta u$ is in bounded subset of $L'_{loc}(\Omega)$ (and $\Omega \subset \mathbb{R}^2$) then for some constant C'

$$\exp \frac{u}{C'} \in L^1(\Omega') \quad (\Omega' \text{ choosen as above})\ .$$

Now if $C > C'$, this implies that $f(u) \in L^{C/c'}_{loc}(\Omega')$. Thus $u \in W^{2,C/c'}_{loc}(\Omega')$ and this implies the L^∞_{loc} bound.

III. The existence result:

Theorem 2: Under the assumptions of Theorem 1 and if, in addition, we assume

$$\overline{\lim_{t\to 0_+}} \frac{f(t)}{t} < \lambda_1\ , \tag{4}$$

then there exists a positive solution of (1); that is, $u(x) > 0$ in Ω and u satisfies

$$-\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad u \in C^2(\bar{\Omega})\ . \tag{1}$$

Remark 3: Actually, Theorem 1 implies (as in [4]) the existence of a branch of positive solutions (λ,u) of the eigenvalue problem

$$-\Delta u = \lambda f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad u \in C^2(\bar{\Omega})\ ,$$

provided f satisfies, instead of (4),

$$\lim_{t\to 0_+} \frac{f(t)}{t} = 0\ . \tag{4'}$$

Remark 4: Theorem 1 can also be used in order to obtain existence results in the case where $f(0) > 0$ (see [6]).

Proof of Theorem 2: We use the same argument as in [4]. To this end we have only to establish the statement (5) below. We can then apply the proof of [4] (which uses

a simple fixed point argument):

$$(5)\quad \begin{cases} \text{If } (t,u) \text{ satisfy } -\Delta u = f(u) + t\sigma_1 \text{ in } \Omega,\ u \in C^2(\bar{\Omega}),\ u \geq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega,\ t \geq 0 \text{ then } t \leq t_0,\ \|u\|_{L^\infty(\Omega)} \leq C_0 \ . \end{cases}$$

where σ_1 is choosen as in the proof of Theorem 1.

To prove (5), we consider u a solution of

$$\begin{cases} -\Delta u = f(u) + t\sigma_1 \text{ in } \Omega \ , \\ u \in C^2(\bar{\Omega}), \quad u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \ , \end{cases}$$

with $t \geq 0$. The first step of the proof of Theorem 1 then shows that we have

$$t \leq t_0.$$

Now, the second step of the proof applies, since the results of [5] still hold for u which is a solution of

$$\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega \ , \\ u \in C^2(\bar{\Omega}), \quad u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \ , \end{cases}$$

with $f(x,u) = f(u) + t\sigma_1(x)$. Indeed we have

$$\frac{\partial\sigma_1}{\partial\nu} \leq -\alpha < 0 \text{ on } \partial\Omega \ ,$$

where ν is the unit exterior normal vector, and this enables us to apply the results and the methods of [5].

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A PRIORI ESTIMATES AND EXISTENCE RESULTS FOR SOME SEMILINEAR ELLIPTIC PROBLEMS

P. L. Lions*

I. Introduction:

In this paper we consider the following problem: find a solution of u

(1) $$\begin{cases} -\Delta u = f(u) & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

where f is a given nonlinear function such that $f(0) = 0$, and Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary. Obviously zero is a solution of (1), so we look for a non-trivial solution of (1), and actually we shall restrict ourselves to positive solutions.

Under various conditions on f, the existence of a positive, non-trivial solution to (1) has been proved, for instance, by P. H. Rabinowitz [9], [10], A. Ambrosetti and P. H. Rabinowitz [8], R. E. L. Turner [11], H. Brézis and R. E. L. Turner [4], R. Nussbaum [7], H. Amann [1], S. I. Pohozaev [8].

We shall be concerned here with the superlinear case i.e., we shall assume that the function f satisfies the condition

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where λ_1 is the first eigenvalue of $-\Delta$, with Dirichlet boundary conditions. One way to solve (1), in this case, is to obtain L^∞ a priori estimates for positive solutions. This was done in [11], [4], assuming that

$$\lim_{t\to\infty} f(t)t^{-\frac{N+1}{N-1}} = 0 .$$

We prove here that such estimates can be obtained if

(3) $$\lim_{t\to\infty} f(t)t^{-\beta} = 0 \quad \text{for some } \beta < \frac{N}{N-2} \quad (\text{if } N = 2 \text{ for any } \beta)$$

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provided Ω (when $N \geq 3$) has some geometrical properties. For example, we proved that there exists a constant C such that if u satisfies

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with $\beta < \frac{N}{N-2}$ and Ω convex, then $\|u\|_{L^\infty(\Omega)} \leq C$.

The main new ingredient used to obtain such estimates is the convenient application of some recent remarkable results of B. Gidas, Wei-Ming Ni and L. Nirenberg [5]. They prove that if u is a positive solution of (1), then the set of critical points of u is contained in a proper subset of Ω (independent of u and f), if Ω (for $N \geq 3$) has some geometrical properties. Using this result, an easy method is then used to give the needed a priori estimates and the existence of positive solutions of (1).

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II. A priori estimates:

Let Ω be a bounded, connected domain with a smooth boundary:

Theorem 1: We assume that f is locally Lipschitz from $\mathbb{R}_+$ to $\mathbb{R}_+$, satisfies (2) and (3). We also assume that if $N \geq 3$, Ω is convex. Then there exists a constant C such that for all $u \in C^2(\bar{\Omega})$ satisfying (1')

$$(1') \qquad \begin{cases} -\Delta u = f(u) \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega; \end{cases}$$

we have

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Remark 1: Instead of assuming Ω convex, we could have assumed

$$(*) \qquad \begin{cases} \exists \Omega' \text{ proper closed subset of } \Omega \text{ such that for all positive } u \text{ satisfying} \\ (1') \text{ the set } \{x \in \Omega \ \ \nabla u(x) = 0\} \subset \Omega'. \end{cases}$$

In view of the results of B. Gidas, Wei-Ming Ni and L. Nirenberg [5], (*) is satisfied for every domain Ω in $\mathbb{R}^2$, and for convex domains in $\mathbb{R}^N$ $(N \geq 3)$. Other cases

are given in [5]. The question of whether (*) is satisfied for any domain in $\mathbb{R}^N$ $(N \geq 3)$ is open and a positive answer would imply that Theorem 1 is true without the restriction Ω convex.

Proof of Theorem 1: The proof is divided into two steps: 1) we prove that u and f are bounded in $L^1_{loc}(\Omega)$, 2) we deduce the $L^\infty(\Omega)$ estimate.

1) Let σ_1 be a fixed eigenfunction of $-\Delta$ corresponding to the first eigenvalue i.e.:

$$-\Delta\sigma_1 = \lambda_1\sigma_1 \quad \text{in } \Omega, \qquad \sigma_1 = 0 \quad \text{on } \partial\Omega, \qquad \sigma_1 \in C^2(\bar{\Omega}) .$$

We may choose σ_1 so that $\sigma_1 > 0$ in Ω. Now (as in [11], [4], [7], [3]) we multiply (1) by σ_1 and we obtain

$$\lambda_1 \int_\Omega u\sigma_1 dx = \int_\Omega f(u)\sigma_1 dx .$$

But using (2) there exists $\lambda > \lambda_1$, $C > 0$ such that $f(t) \geq \lambda t - C$ for every $t \geq 0$. Thus we deduce $\int_\Omega u\sigma_1 \leq$ const., and $\int_\Omega f(u)\sigma_1 dx \leq$ const. Since $\sigma_1 > 0$ in Ω, we have an estimation of u and $f(u)$ in $L^1_{loc}(\Omega)$.

2) Thus, $-\Delta u$ and u are in a bounded subset of $L^1_{loc}(\Omega)$. But this implies by standard regularity results (localize and use $L^1 \subset W^{-\epsilon,p} \forall p < \frac{N}{N-\epsilon}$) that u is bounded in $L^p_{loc}(\Omega)$ $(\forall p < \frac{N}{N-2}$ if $N \geq 3$, $\forall p < \infty$ if $N = 2)$. Now we use (3) and we have

$$f(u) \text{ is bounded in } L^{q_1}_{loc}(\Omega) \text{ with } q_1 > 1 .$$

This implies u is bounded in $W^{2,q_1}_{loc}(\Omega)$ and by a standard bootstrap argument we obtain after a finite number of iterations of the preceding argument

$$u \text{ is bounded in } W^{2,q}_{loc}(\Omega) \text{ with } q > \frac{N}{2} ,$$

in particular u is bounded in $L^\infty_{loc}(\Omega)$. To conclude, we apply the previously mentioned result of [5]: indeed, there exists some open set Ω' (independent of f and u) such that if $u \neq 0$

$$\{x \in \Omega \ \nabla u(x) = 0\} \subset \Omega' \subset \bar{\Omega}' \subset \Omega .$$

Thus, $\|u\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(\Omega')}$, and $\|u\|_{L^\infty(\Omega')} \leq$ const.

Remark 2: 1) If $N = 2$, with a slight refinement of the second part of the proof, we can prove Theorem 1 with the assumption

$$(3') \qquad \lim_{t\to\infty} f(t)\exp(-t/C) = 0 ,$$

where C is some constant depending only on Ω, and on $\lim_{t\to\infty} \frac{f(t)}{t}$. Indeed, if u is such that $-\Delta u$ is in bounded subset of $L^1_{loc}(\Omega)$ (and $\Omega \subset \mathbb{R}^2$) then for some constant C'

$$\exp \frac{u}{C'} \in L^1(\Omega') \quad (\Omega' \text{ choosen as above}) .$$

Now if $C > C'$, this implies that $f(u) \in L^{C/c'}_{loc}(\Omega')$. Thus $u \in W^{2,C/c'}_{loc}(\Omega')$ and this implies the L^∞_{loc} bound.

III. The existence result:

Theorem 2: Under the assumptions of Theorem 1 and if, in addition, we assume

$$(4) \qquad \overline{\lim_{t\to 0_+}} \frac{f(t)}{t} < \lambda_1 ,$$

then there exists a positive solution of (1); that is, $u(x) > 0$ in Ω and u satisfies

$$(1) \qquad -\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad u \in C^2(\bar{\Omega}) .$$

Remark 3: Actually, Theorem 1 implies (as in [4]) the existence of a branch of positive solutions (λ,u) of the eigenvalue problem

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provided f satisfies, instead of (4),

$$(4') \qquad \lim_{t\to 0_+} \frac{f(t)}{t} = 0 .$$

Remark 4: Theorem 1 can also be used in order to obtain existence results in the case where $f(0) > 0$ (see [6]).

Proof of Theorem 2: We use the same argument as in [4]. To this end we have only to establish the statement (5) below. We can then apply the proof of [4] (which uses

a simple fixed point argument):

$$(5) \quad \begin{cases} \text{If } (t,u) \text{ satisfy } -\Delta u = f(u) + t\sigma_1 \text{ in } \Omega,\ u \in C^2(\bar{\Omega}),\ u \geq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega,\ t \geq 0 \text{ then } t \leq t_0,\ \|u\|_{L^\infty(\Omega)} \leq C_0 . \end{cases}$$

where σ_1 is choosen as in the proof of Theorem 1.

To prove (5), we consider u a solution of

$$\begin{cases} -\Delta u = f(u) + t\sigma_1 \text{ in } \Omega , \\ u \in C^2(\bar{\Omega}), \quad u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega , \end{cases}$$

with $t \geq 0$. The first step of the proof of Theorem 1 then shows that we have

$$t \leq t_0 .$$

Now, the second step of the proof applies, since the results of [5] still hold for u which is a solution of

$$\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega , \\ u \in C^2(\bar{\Omega}), \quad u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega , \end{cases}$$

with $f(x,u) = f(u) + t\sigma_1(x)$. Indeed we have

$$\frac{\partial\sigma_1}{\partial\nu} \leq -\alpha < 0 \text{ on } \partial\Omega ,$$

where ν is the unit exterior normal vector, and this enables us to apply the results and the methods of [5].

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We study the existence of non-trivial, positive solutions of

$$(1) \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We prove the existence of a priori estimates for positive solutions of (1), provided f does not increase too rapidly at infinity and provided Ω has some geometrical properties. From these a priori estimates we deduce the existence of non-trivial, positive solutions of (1).

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